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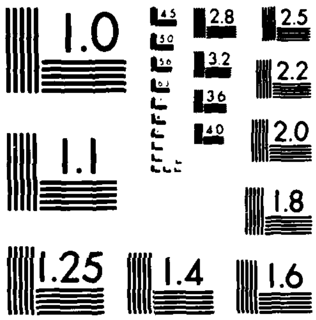

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ON LARGE MATCHINGS AND

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by

A. M. Frieze*

January 1984

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Carnegie-Mellon University

PITTSBURGH, PENNSYLVANIA 15213

GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

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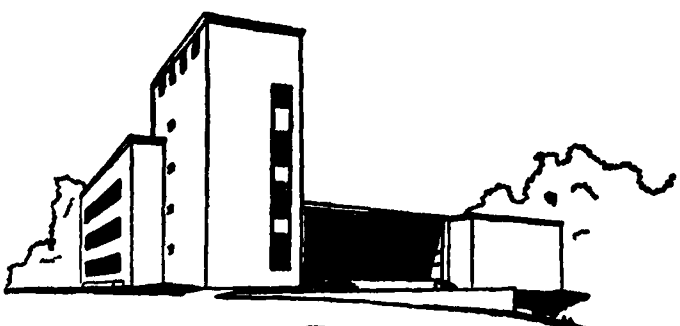

Management Science Research Report No. MSRR 504

ON LARGE MATCHINGS AND CYCLES IN SPARSE RANDOM GRAPHS

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Abstract

Let $p = c/n$ where c is a large constant. We show that the random graph $G_{n,p}$ a.s. contains a matching of size $n(1 - (1+\varepsilon(c))e^{-c})/2$ and a cycle of size $n(1-(1+\varepsilon(c))ce^{-c})$ where $\varepsilon(c)$ is some function satisfying $\lim_{c\to\infty} \varepsilon(c) = 0$.

1. In this paper Studied ~~we study~~ the size of the largest matching and cycle in random graphs with edge probability c/n where c is a large constant. We continue the analysis of Bollobás [2], Bollobás, Fenner and Frieze [3] and confirm the conjecture in the final paragraph of the latter paper.

We shall let $G_{n,p}$ denote a random graph with vertex set $V_n = \{1,2,\ldots,n\}$ in which edges are chosen independently with probability p. We say that $G_{n,p}$ has a property Q <u>almost surely</u> (a.s.) if $\lim_{n\to\infty} \Pr(G_{n,p} \in Q) = 1$.

For $c > 0$ define $\alpha(c)$, $\beta(c)$ by

$$\text{(1.1)} \qquad \alpha(c) = \sup(\alpha \geq 0:\ G_{n,c/n} \text{ a.s. contains a matching of size at least } \alpha n/2)$$

and

$$\text{(1.2)} \qquad \beta(c) = \sup(\beta \geq 0:\ G_{n,c/n} \text{ a.s. contains a cycle of size at least } \beta n).$$

Our main result is an improved estimate of $\beta(c)$. However the same methods can be used to estimate $\alpha(c)$ and we shall do this first as the analysis is marginally simpler.

In what follows $p = c/n$ and $\varepsilon_1(c)$, $\varepsilon_2(c)$ are unspecified functions satisfying $\lim_{c\to\infty} \varepsilon_i(c) = 0$, $i=1,2$.

<u>Theorem 1.1</u>

$$\text{(1.3)} \qquad \alpha(c) = 1 - (1 + \varepsilon_1(c))e^{-c}$$

and this remains valid if $c\to\infty$.

As far as we know the only other paper dealing with this question is by Karp and Sipser [7] who prove some strong results about a simple heuristic for finding a large cardinality matching.

There has been more work done on estimating $\beta(c)$. Ajtai, Komlós and Szemerédi [1] and Fernandez de la Vega [6] showed that $\beta(c) \geq 1 - c_0/c$. Bollobás made a significant step forward by showing that $G_{n,p}$ a.s. contains a large Hamiltonian subgraph and that $\beta(c) \geq 1 - c^{24}e^{-c/2}$. By refining this analysis, Bollobás, Fenner and Frieze [3] showed that $\beta(c) \geq 1 - c^6 e^{-c}$. The main result of this paper is

Theorem 1.2

$$\beta(c) = 1 - (1+\varepsilon_2(c))\, ce^{-c} \tag{1.4}$$

and this remains valid if $c \to \infty$.

Corollary 1.3

A random digraph with edge density c/n a.s. contains a directed cycle of size $n(1 - (1+\varepsilon_2(c))ce^{-c})$.

Notation

The following notation is used throughout. Let G be a graph. V(G), E(G) denote the sets of vertices and edges of G.

For $S \subseteq V(G)$ we let $G[S] = (S, E(S))$ where $E(S) = \{e \in E(G): e \subseteq S\}$.

$N_G(S) = \{w \in S:$ there exists $v \in S$ such that $\{v,w\} \in E(G)\}$.

For $v \in V(G)$ we write $N_G(v)$ for $N_G(\{v\})$ and $d_G(v)$ for the degree of v. $\mu(G)$ is the maximum cardinality of a matching of G.

$$BS(x,m) = \sum_{k=0}^{\lfloor x \rfloor} \binom{m}{k} p^k (1-p)^{m-k}$$

As the case $c > \log n$ is well known we shall assume for convenience that $ce \leq 3\log n$.

2. Lemma 2.1

Let $G = G_{n,p}$ and let vertex v be small if $d_G(v) \le c/10$ and large otherwise. Let SMALL, LARGE be the sets of small and large vertices respectively.

Let $W = W_1 \cup W_2$ where for k=1,2

$W_k = \{v$: v is small and there exists a small w such that v and w are joined by a path of length $k\}$

Then for $c \ge 300$ G a.s. satisfies the following:

(2.1) $|\{v \in V_n: d_G(v) \le c/10 + 1\}| \le ne^{-2c/3}$;

(2.2) there does not exist $S \subseteq V_n$ with $|S| \ge ne^{-c}$ and $|\{e \in E(G): e \cap S \ne \emptyset\}| \ge 4c\ |S|$;

(2.3) $d_G(v) \le 4\log n$ for $v \in V_n$;

(2.4) $|W| \le c^2 e^{-4c/3} n$;

(2.5) $\emptyset \ne S \subseteq V_n$, $|S| \le n/14$ and $S \subseteq$ LARGE implies $|N_{G(S)}| \ge 6\ |S|$;

(2.6) $S \subseteq V_n$, $n/14 \le |S| \le n/2$ implies $|\{\{v,w\} \in E(G) : v \in S, w \in S\} \ge c\ |S|/10$;

Proof

To prove (2.1) note that for n large

$\mathrm{Exp}(\ |\{v \in V_n: d_G(v) \le c/10 + 1\}\ ) = n\ BS(c/10 + 1, n-1) \le ne^{-.669c}$.

Now the variance of this set size can be shown to be $\le ne^{-2c/3}$.

Thus one can use either the Chebycheff or Markov inequality depending on whether or not c remains bounded as n tends to infinity.

Next note that the probability there exists a set S violating (2.2) is no more than

$$\sum_{s \ge ne^{-c}} \binom{n}{s}\binom{sn}{|4cs|}p^{|4cs|}$$

$$\le \sum_{s \ge ne^{-c}} \left(\frac{ne}{s}\right)^{s} \left(\frac{snep}{4cs}\right)^{4cs}$$

$$\le \sum_{s \ge ne^{-c}} \left(\frac{e^{5+1/c}}{256}\right)^{cs} = o(1).$$

To prove (2.3) we observe that

$$\mathrm{Exp}(|\{v \in V_n : d_G(v) > 4\log n\}|) = n \sum_{k>4\log n} \binom{n-1}{k} p^k (1-p)^{n-k-1}$$

$$\le n \sum_{k>4\log n} \left(\frac{ce}{k}\right)^{k} = o(1)$$

as $ce \le 3\log n$.

Next let P_k = {paths of length k in G with small endpoints} . Now clearly

(2.7) $\quad |W_k| \le 2\,|P_k| \qquad$ for k=1,2.

Furthermore

(2.8) $\quad \mathrm{Exp}(|P_1|) = \binom{n}{2}p\lambda^2$

where $\lambda = BS(c/10 - 1, n-2) \le e^{-.669c}$

Now

$$\mathrm{Exp}(|P_1|^2) = \mathrm{Exp}(|P_1|) + \binom{n}{2}\binom{n-2}{2}p^2\lambda_1 + 2(n-2)\binom{n}{2}p^2\lambda_2$$

where

$$\lambda_1 = \Pr(\mathrm{SMALL} \supseteq \{1,2,3,4\} \mid E(G) \supseteq \{\{1,2\}, \{3,4\}\})$$
$$\le \Pr(|N_G(1) \cap \{5,6,\ldots,n\}| \le c/10 - 1)^4$$
$$\le (\lambda(1-p)^{-2})^4$$

and

$$\lambda_2 = \Pr(\mathrm{SMALL} \supseteq \{1,2,3\} \mid E(G) \supseteq \{\{1,2\}, \{2,3\}\})$$
$$\le (\lambda(1-p)^{-1})^3.$$

This gives

(2.9) $\quad \mathrm{Var}(|P_1|) \le ce^{-4c/3}n \qquad$ for n large.

Similar calculations give

(2.10a) $\quad \mathrm{Exp}(|E_2|) = (1+o(1))n^3p^2\lambda^2/2$

and

(2.10b) $\quad \mathrm{Var}(|E_2|) \le n^3p^2\lambda^2 \qquad$ for n large

(2.4) now follows from (2.7), (2.8), (2.9) and (2.10).

To prove (2.5) we first consider S for which $1 \le s = |S| \le n/35000e^4$. Let $T = S \cup N_G(S)$ and $t = |T|$. If (2.5) does not hold for S then $|T| \le m_1 = \lceil n/5000e^4 \rceil$ and T contains at least $m_2 = \lceil ct/140 \rceil$ edges of G. The probability that such a T exists is no more than

$$\sum_{t=1}^{m_1} \binom{n}{t} \binom{\binom{t}{2}}{m_2} p^{m_2} \le \sum_{t=1}^{m_1} \left(\frac{ne}{t}\right)^t \left(\frac{t^2ep}{2m_2}\right)^{m_2}$$

$$\le \sum_{t=1}^{m_1} \left(\frac{ne}{t}\right)^t \left(\frac{70et}{n}\right)^{ct/140} \le \sum_{t=1}^{m_1} \left(\frac{4900e^4t}{n}\right)^{ct/280} = o(1)$$

using $c \ge 300$.

For $|S| \ge m_3 = \lceil n/36000e^4 \rceil$ we can ignore the fact that the vertices of S are large. The probability that such an S exists violating (2.5) is no more than

$$\sum_{s = m_3}^{\lfloor n/14 \rfloor} \binom{n}{s}\binom{n}{6s}(1-p)^{s(n-7s)}$$

$$\leq \sum_{s = m_3}^{\lfloor n/14 \rfloor} \left(\frac{ne}{s}\right)^{s} \left(\frac{ne}{6s}\right)^{6s} e^{-cs/2}$$

$$\leq \sum_{s = m_3}^{\lfloor n/14 \rfloor} \left(6^{8} \cdot 10^{21} \cdot e^{35} \cdot e^{-c/2}\right)^{s} = o(1)$$

which proves (2.5).

The probability that (2.6) does not hold is not more than

$$\sum_{s=\lceil n/14 \rceil}^{\lfloor n/2 \rfloor} \binom{n}{s} BS(cs/10,\ s(n-s))$$

$$\leq 2 \sum_{s=\lceil n/14 \rceil}^{\lfloor n/2 \rfloor} \left(\frac{ne}{s}\right)^{s} \left(\frac{10s(n-s)e}{cs}\right)^{cs/10} \left(\frac{c}{n}\right)^{cs/10} e^{-cs/3}$$

$$\leq 2 \sum_{s=\lceil n/14 \rceil}^{\lfloor n/2 \rfloor} \left(14e(10e)^{c/10} e^{-c/3}\right)^{s} = o(1).$$

∎

The proofs of our theorems rely on the removal of a certain set of vertices. We must show that this set is not too large. The following Lemma deals with part of this set.

Lemma 2.2

Let X_0 = SMALL and let the sequence of sets $X_1, X_2, \ldots, X_s$ be defined by

$$X_i = \{v \in V_n : |N_G(v) \cap \bigcup_{t=0}^{i-1} X_t| \geq 2\}$$

and let s be the smallest $i \geq 1$ such that $X_{i+1} = X_i$. Let $X = \bigcup_{i=1}^{s} X_i$, then

(2.11) $\quad |X| \leq 2e^4c^4e^{-4c/3}n \qquad$ a.s.

Proof

For $x \in X \cup X_0$ let $i(x)=\min\{i: x \in X_i\}$ and let $D(x)=(V(x), A(x))$ denote a digraph inductively constructed as follows: for $x \in X_0$, $D(x) = (\{x\},\emptyset)$ and for $x \in X_0$ let y_1, y_2 be 2 distinct neighbours of x satisfying $i(x) > i(y_1), i(y_2)$. Then

$$D(x) = (V(y_1) \cup V(y_2) \cup \{x\}, A(y_1) \cup A(y_2) \cup \{(x, y_1), (x, y_2)\})$$

Each $D(x)$ is acyclic, (weakly) connected and satisfies

(2.12) each $v \in V(x)$ has outdegree 0 or 2 and x is the unique vertex of indegree 0.

Let

k = the number of vertices of outdegree 2 = $|K(x)|$, where $K(x)=S(x)-X_0$.

and let

ℓ = the number of vertices of outdegree 0 = $|L(x)|$, where $L(x)=S(x) \cap X_0$.

It follows then that

(2.13a) $|A(x)| = 2k$

and we will show

(2.13b) $\ell \leq k+1$ and if $\ell=k+1$ then $D(x)$ is a binary tree rooted at x.

This is most easily proved by induction on k. A digraph satisfying (2.12) has at least one vertex y whose outneighbours z_1, z_2 both have outdegree zero. Removing arcs (y, z_1) and (y, z_2) and any vertex which becomes isolated we obtain a smaller digraph satisfying (2.12).

We obtain from the above that we can associate with each $x \in X$, a set $V(x)$ of vertices and a partition of $V(x)$ into $K(x)$, $L(x)$ satisfying

(2.14a) $x \neq x'$ implies $V(x) \neq V(x')$;

(2.14b) if $k = |K(x)|$, $\ell = |L(x)|$ then $2 \leq \ell \leq k+1$;

(2.14c) $L(x) \subseteq$ SMALL;

(2.14d) $G(x)=G[V(x)]$ is connected and has at least $2k$ edges;

(2.14e) if $\ell=k+1$ and $G(x)$ has $2k$ edges then $G(x)$ is a tree with leaves $L(x)$.

We estimate $|X_S - X_0|$ by counting sets of vertices satisfying (2.14). For a given k, ℓ, m let $\lambda_{k,\ell,m}$ be the expected number of sets K, L with $|K|=k$, $|L| =\ell$ satisfying (2.14) above, where $G[K \cup L]$ has m edges. Then

$$\lambda_{k,\ell,m} \leq \binom{n}{k}\binom{n}{\ell}\binom{\binom{k+\ell}{2}}{m} p^m BS(c/10,\ n-k-\ell)^{\ell}$$

$$\leq \left(\frac{ne}{k}\right)^k \left(\frac{ne}{\ell}\right)^{\ell} \left(\frac{(k+\ell)^2 e}{2m}\right)^m \left(\frac{c}{n}\right)^m e^{-2c\ell/3}\left(1 - \frac{c}{n}\right)^{-\ell(k+\ell)}$$

$$= \mu_{k,\ell,m}$$

Now if $c \leq 2\log n$, $k, \ell \leq n^{1/3}$ then $\mu_{k,\ell,m+1}/\mu_{k,\ell,m} \leq n^{-1/4}$ for n large. Thus

$$\sum_{m=2k}^{\binom{k+\ell}{2}} \lambda_{k,\ell,m} \leq (1+o(1))\, \mu_{k,\ell,2k}\ . \tag{2.15}$$

With the same bounds on c,k,ℓ and with n large and $\ell \leq k+1$ we have

$$\mu_{k,\ell,2k} \leq 21 n^{\ell-k}(e^4c^2k)^k \ell^{-\ell} e^{-2c\ell/3} \tag{2.16}$$ which implies

$$\sum_{\ell=2}^{k+1} \mu_{k,\ell,2k} \leq 21(e^4c^2k/n)^k \sum_{\ell=2}^{k+1} (n/\ell e^{2c/3})^{\ell}$$

$$\le n(e^4c^2)^k e^{-2ck/3}$$

$$\le ne^{-ck/2} \qquad \text{as } c \ge 300.$$

It follows that $s \le \log n$ a.s., and we can assume $k \le \log n$. Now, using (2.16),

$$\sum_{k=2}^{\log n} \sum_{\ell=2}^{k} \mu_{k,\ell,2k} \le 21 \sum_{k=2}^{\log n} (e^4c^2)^k e^{-2ck/3}$$

$$\le 22(e^4c^2)^4 e^{-4c/3}$$

and so

(2.17) the number of sets K, L with $2 \le \ell \le k$ is a.s. less than $n^{1/2}e^{-4c/3}$.

We only need to consider the case $\ell=k+1$ from now on. But as $\mu_{k,k+1,m+1}/\mu_{k,k+1,m} \le 3ck/n$ we have

$$(2.18) \qquad \sum_{m \ge 2k} \mu_{k,k+1,m} \le (1+o(1))\mu_{k,k+1,2k}$$

So we are finally reduced to estimating

τ_k = the number of <u>vertex induced</u> binary trees with k leaves (<u>k-b-trees</u>) in which each leaf is small.

Let θ_k be the number of (vertex labelled) k-b-trees contained in a complete graph with 2k-1 vertices. (Clearly $\theta_k \le (2k-1)^{2k-3}$). Then

$$(2.19) \quad \mathrm{Exp}(\tau_k) = \binom{n}{2k-1}\theta_k p^{2k-2}(1-p)^{\binom{2k-1}{2}-2k+2} BS(c/10-1,n-2k+1)^k$$

$$\le n(e^2c^2e^{-2c/3})^k \qquad \text{for } n \text{ large.}$$

To estimate $Var(\tau_k)$, let $\{T_1, T_2, \ldots, T_B\}$, $B = \binom{n}{2k-1}\theta_k$, be the set of k-b-trees contained in a complete graph with n vertices. Let A_i be the event that T_i is a vertex induced subgraph of G_p in which all leaves are small.

Next let $Y_p = \{(i,j): |V(T_i) \cup V(T_j)| = p\}$ for $p=2k-1,\ldots,4k-2$ and let $Z_{p,q} = \{(i,j) \in Y_p : |E(T_i) \cup E(T_j)| = q\}$. Then

$$\text{(2.20)} \qquad Exp(\tau_k^2) = Exp(\tau_k) + \Delta_1 + \Delta_2$$

where

$$\Delta_1 = \sum_{(i,j) \in Y_{4k-2}} Pr(A_i \cap A_j)$$

and

$$\Delta_2 = \sum_{p=2k-1}^{4k-3} \sum_{(i,j) \in Y_p} Pr(A_i \cap A_j)$$

Now

$$\Delta_1 \le \binom{n}{2k-1}^2 \left(\theta_k p^{2k-2} (1-p)^{\binom{2k-1}{2}-2k+2}\right)^2 \sigma$$

where

$$\sigma = BS(c/10-1,\ n-2k+1)^k\ BS(c/10-1,\ n-4k+2)^k$$

is an estimate of the probability that all leaves of 2 particular disjoint trees are small.

It follows that

$$\text{(2.21)} \qquad \Delta_1 \le Exp(\tau_k)^2 (1-p)^{-2k^2}$$

Now for $p \le 4k-3$ we have

$$\sum_{(i,j)\in Y_p} \Pr(A_i \cap A_j) = \sum_{q=p-1}^{4k-4} \sum_{(i,j)\in Z_{p,q}} \Pr(A_i \cap A_j)$$

$$\le \sum_{q=p-1}^{4k-4} \binom{n}{p}\binom{\binom{p}{2}}{q}\binom{q}{2k-1}^2\left(\frac{c}{n}\right)^q e^{-2ck/3}(1-p)^{-8k^2}$$

(2.22) $\qquad \le ne^{-ck/2}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ for n large.

(2.19), (2.20), (2.21), (2.22) plus the Chebycheff inequality implies that τ_k is a.s. within a factor $(1+o(1))$ of the R.H.S. of (2.19). This together with (2.17) and (2.18) proves the result.

∎

For a positive integer k, the <u>k-core</u> $V_k(G)$ is defined to be the largest set $S \subseteq V_n$ such that $\delta(G[S]) \ge k$. This is well defined, for if $\delta(G[S_i]) \ge k$ for $i=1,2$ then $\delta(G[S_1 \cup S_2]) \ge k$. We let G_k denote the subgraph of G induced by $V_k(G)$.

The k-core can be constructed using the following algorithm:

```
begin
    H:=G;
    while δ(H) < k do
    begin
        Y: = {v ε V(H) : d_H(v) < k};
        H: = H[V(H) - Y]
    end
end
```

On termination $H=G_k$. This is because one can easily show inductively that each iteration removes vertices that are not in $V_k(G)$ and as $\delta(H) \geq k$ we have $V(H) \subseteq V_k(G)$.

Clearly any matching of G is contained in G_1(= G minus isolated vertices) and any cycle of G is contained in G_2.

Now for k=1,2 let $A_k = A_k(G_{n,p}) = V_k(G_{n,p})-(W \cup X \cup Y_k)$ where W,X are as defined in Lemmas 2.1, 2.2 respectively and

$$Y_k = \{y \in V_n : d_{G_{n,p}}(y) = k \text{ and } N_{G_{n,p}}(y) \cap X \neq \emptyset\}.$$

Let $H_k = H_k(G_{n,p}) = G_{n,p}[A_k]$, then we have

Lemma 2.3

For k=1,2 let M be any matching of $G_{n,p}[A_k]$ which is not incident with any small vertex. Let $\hat{H}_k=H_k-M$, then (2.5) implies:

(2.23) $\quad \emptyset \neq S \subseteq A_k,\ |S| \leq n/14$ implies $|N_{\hat{H}_k}(S)| \geq k|S|$.

Proof

Let $G=G_{n,p}$, $H=\hat{H}_k$ and for a given S let $S_1 = S \cap \text{SMALL}$ and $S_2 = S-S_1$. Now

(2.24) $\quad |N_H(S)| \geq |N_H(S_1)| - |S_2| + |N_H(S_2)| - \min(\ |S_1|, |S_2|\ )$

We can write $\min(\ |S_1|,\ |S_2|\ )$ in place of $|S_1|$ as no vertex of S_2 is adjacent to more than one vertex of S_1, as $S_2 \cap X = \emptyset$.

Also, we claim

(2.25) $\quad |N_H\ (S_1)| \geq k|S_1|$.

Note first that $v \in S_1$ implies $d_{G_k}(v) \geq k$ and no pair of vertices of S_1 are adjacent, since $S_1 \cap W_1 = \emptyset$. Note that no pair of vertices of S_1 have a common neighbour as $S_1 \cap W_2 = \emptyset$. Also $N_G(S_1) \cap (W \cup Y_k) = \emptyset$ as

$S_1 \cap W_1 = \emptyset$. Furthermore $v \in S_1$ implies $|N_G(v) \cap X| \leq 1$ as $S_1 \cap X = \emptyset$. Thus to prove (2.25) we need only show that if $v \in S_1$ and $d_G(v)=k$ then $N_{G(v)} \cap X = \emptyset$. But this follows from $S_1 \cap Y_k = \emptyset$.

We claim next that if (2.5) holds then

(2.26) $|N_H(S_2)| \geq 4|S_2|$

For then $|N_G(S_2)| \geq 6|S_2|$ and for each $v \in S_2$, $|N_G(v)| \leq |N_H(v)| + 2$. This is because v is incident with at most one edge of M and is adjacent to at most one vertex of W X Y_k. It is a simple matter to verify (2.23) from (2.24), (2.25) and (2.26) by considering $|S_1| \geq |S_2|$ and $|S_1| < |S_2|$ as separate cases.

∎

3. Matchings

Let H_1 be the subgraph of G defined in Lemma 2.3. We are going to prove that H_1 a.s. has a perfect or near perfect matching. We first establish that H_1 is large.

Lemma 3.1

(3.1) $\quad |V(H_1)| = n(1 - (1+\varepsilon_1(c))e^{-c}) \qquad$ a.s.

where $\varepsilon_1(c) \to 0$ as $c \to \infty$.

Proof

$$|V(H_1)| \geq |V_1(G)| - |W| - |X| - |Y_1-W| .$$

It is well known that

(3.2) $\quad |V_1(G)| = (1+o(1))n(1-e^{-c}) \qquad$ a.s.

where the o(1) term in (3.2) could for example be taken to be $\pm n^{-1/4}e^{-c/2}$, using the Chebycheff inequality.

Lemmas 2.1 and 2.2 give a.s. upper bounds on $|W|$, $|X|$ and (3.1) will follow from

(3.3) $\quad |Y_1-W| \leq |X|$

For $y \in Y_1$ there is, by definition, a unique $x(y) \in X$ such that y is adjacent to $x(y)$ in G. Now for distinct y_1, y_2 $\in Y_1 - W$ we have $x(y_1) \neq x(y_2)$ else $y_1 \in W_2$ and (3.4) follows.

We establish next the following condition that goes with a graph not having a (near) perfect matching.

Lemma 3.2

Suppose $\mu(H) < \lfloor |V(H)|/2 \rfloor$. Let $\mathfrak{M}$ be the set of maximum cardinality matchings of H. Let $U=\{u_1, u_2,\ldots,u_t\}$ be the set of vertices left isolated by some $M \in \mathfrak{M}$. For i=1,2,...,t there exists a set $U_i \subseteq U$ satisfying

(3.4a) $|N_H(U_i)| < |U_i|$;

(3.4b) $w \in U_i$ implies $e=\{u_i,w\} \notin E(H)$ and $\mu(H) < \mu(H+e)$.

Proof

Let $u_i \in U$ and let some $M_i \in \mathfrak{M}$ leave u_i isolated. Let $S_i \neq \emptyset$ be the set of vertices, different from u_i, left isolated by M_i. Let U_i' be the set of vertices reachable from S_i be an even length alternating path w.r.t. M_i. Let $U_i = S_i \cup U_i' \subseteq U$. Then (3.4b) holds otherwise M_i has an augmenting path.

If $u \in N_H(U_i)$ then $u \notin S_i$ and so there exists y_1 such that $\{u,y_1\} \in M_i$. We show that $y_1 \in U_i$ which will prove (3.4a). Now there exists $y_2 \in U_i$ such that $\{u,y_2\} \in E(H)$. Let P be an even length alternating path from some $s \in S_i$ terminating at y_2. If P contains $\{u,y_1\}$ we can truncate it to terminate with $\{u,y_1\}$, otherwise we can extend it using edges $\{y_2,x\}$ and $\{x,y_1\}$.

We are now ready for the

Proof of Theorem 1.1

We use a coloring argument that was introduced in Fenner and Frieze [5]. Suppose that after generating $G=G_{n,p}$ all its edges are colored blue, and then each edge of G is re-colored green with probability p'=logn/cn and left blue with probability 1-p'. These recolourings are done independently of each

other.

Let E^b, E^g denote the blue and green edges respectively and let $G^b = (V_n, E^b)$, $H_1 = H_1(G)$ and $H_1^b = H_1(G^b)$.

Remark 3.1

It is important to note that for a fixed value of E^b, E^g is a random subset of $\overline{E}^b$ where each $e \in \overline{E}^b$ is independently included in E^g with probability $p_1 = pp'/(1-p(1-p'))$ and excluded with probability $1-p_1$.

Consider next the following 2 events:

$\mathcal{G} \equiv G = G_{n,p}$ satisfies the conditions of Lemmas 2.1, 2.2 and $\mu(H_1) < \lfloor |V(H_1)| \rfloor /2$.

$\mathcal{E} \equiv$ (a) $\emptyset \neq S \subseteq A_1(G^b)$, $|S| \leq n/14$ implies $|N_{H_1^b}(S)| \geq |S|$;

(b) $\mu(H_1^b) < \lfloor |V(H_1^b)|/2 \rfloor$;

(c) there does not exist $e = \{v,w\} \in E^g$, $e \subseteq A_1(G^b)$ such that some maximum cardinality matching of H_1^b leaves both v and w isolated.

In consequence of what has already been proved, we need only prove

(3.5) $\lim_{n\to\infty} \Pr(\mathcal{G}) = 0.$

To prove (3.5) we shall prove

(3.6a) $\Pr(\mathcal{E} \mid \mathcal{G}) \geq (1 - o(1))(1-p')^{2n/3}$

(3.6b) $\Pr(\mathcal{E}) \leq (1-p_1)^{n^2/392}$

which together imply (3.5).

Proof of (3.6a)

Let $G_0 \in \mathcal{G}$ be fixed and let M_0 be any fixed maximum cardinality matching of H_1. We prove

(3.7) $\Pr(\mathcal{E} \mid G_{n,p} = G_0) \geq (1-p')^{2n/3} - 16(\log n)^4/c^2n.$

We can readily verify this once we have shown that

(3.8) $\mathcal{E} \cap \mathcal{G} \supseteq \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \cap \mathcal{G}$

where

$\mathcal{E}_1 \equiv E^g$ is a matching of G_0;

$\mathcal{E}_2$ = no green edge meets any vertex of degree less than $c/10+2$ in G_0 or any vertex in $W \times Y_1$

$\mathcal{E}_3 = M_0 \cap E^g = \emptyset$

For $\mathcal{E}_1 \cap \mathcal{E}_2$ implies

(3.9) $A_1(G_0^b) = A_1(G_0)$

and then $\mathcal{E}_1$ implies (see Lemma 2.3) that (2.23) holds, which verifies $\mathcal{E}(a)$. $\mathcal{E}(b)$ follows directly from (3.9) and $G_0 \in \mathcal{G}$. $\mathcal{E}_3$ implies $\mu(H_1^b) = \mu(H_1)$ and $\mathcal{E}(c)$.

Now it follows from (2.3) that

(3.10) $\Pr(\bar{\mathcal{E}}_1) \leq 16(\log n)^4/c^2n.$

From Lemmas 2.1, 2.2 and (3.3) we find that the total number of edges of G_0 that are excluded by the conditions in $\mathcal{E}_2$, $\mathcal{E}_3$ is no more than

$$n((c/10 + 1)e^{-2c/3} + 4nce^{-ce})n + n/2 \leq 2n/3$$

Thus

$$\Pr(\bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \cup \bar{\mathcal{E}}_3) \leq 1-(1-p')^{2n/3} + 16(\log n)^4/c^2n$$

which proves (3.7).

Proof of (3.6b)

Now

(3.11) $\Pr(\mathcal{E}) = \sum_{\Gamma} \Pr(\mathcal{E} \mid G^b = \Gamma)\Pr(G^b = \Gamma)$

where Γ is an arbitrary graph with vertices V_n.

Now if $H_1(\Gamma)$ fails to satisfy $\mathcal{E}(a)$, $\mathcal{E}(b)$ then $\Pr(\mathcal{E}|G^b = \Gamma) = 0$. So let us assume that $\mathcal{E}(a)$, $\mathcal{E}(b)$ hold.

Now if $U, U_1, \ldots, U_t$ are as defined in Lemma 3.2 with $H=H_1$, then each set is of size at least $n/14$ and for $\mathcal{E}(c)$ to hold no green edge can join $u_i \in U$ to $w \in U_i$. But then in view of Remark 3.1 and $\mathcal{E}(a)$ we have

$$\Pr(\mathcal{E}(c) \mid G^b = \Gamma) \le (1-p_1)^{n^2/392}$$

which implies (3.6b).

We have thus shown that

$$\mu(G) \ge n(1 - (1+\varepsilon_1(c))e^{-c})/2 \qquad \text{a.s.}$$

On the other hand (3.2) implies

$$\mu(G) \le n(1+o(1))(1 - e^{-c})/2 \qquad \text{a.s.}$$

and Theorem 1.1 follows.

■

If we put $c=\log n + \omega$ where $\omega \to \infty$ then we have $\alpha(c)=1-(1+o(1))e^{-\omega}n^{-1}$ and then $G_{n,p}$ a.s. has a matching of size at least $(n - (1+o(1))e^{-\omega})/2$. This is Erdos and Rènyi's result [4], (what we have proved is that H_1 a.s. has a matching of size $\lfloor |V(H_1)|/2 \rfloor$ and one can see that when $c=\log n+\omega$, $H_1=G_{n,p}$ a.s.).

4. Cycles

Let H_2 be the subgraph of G defined in Lemma 2.3. We are going to prove that H_2 a.s. has a hamiltonian cycle. The proof is very similar to that of section 3 and as such we will only give the essential differences.

Lemma 4.1

(4.1) $\quad |V(H_2)| = n(1 - (1+\varepsilon_2(c))ce^{-c}) \qquad$ a.s.

where $\varepsilon_2(c) \to 0$ as $n \to \infty$

Proof

$$|V(H_2)| \geq |V_2(G)| - |W| - |X| - |Y_2 - W \cup X|$$

Now

$$|Y_2 - W \cup X| \leq |X|$$

follows by a similar argument to (3.3). Now let Z_0 be the set of vertices of degree 0 or 1 in G and let Z_1, $Z_2, \ldots$ be the sequence of sets removed in each iteration of the 2-core finding algorithm of section 2. Now, corresponding to (3.2), it is also well known that

$$Z_0 = (1-o(1))n(1-ce^{-c}) \qquad \text{a.s.}$$

We complete the proof of the lemma by showing that

$$Z_i \subseteq X \cup W_1 \cup Y_2 \qquad i=1,2,\ldots$$

Thus assume inductively that $Z_1, Z_2, \ldots Z_{i-1} \subseteq X \cup W_1 \cup Y_2$ for some i ≥ 1 (true vacuously for i=1) and let $T = \bigcup_{t=0}^{i-1} Z_t$.

Then $y \in Z_i$ implies $d_G(y) \geq 2$ but $|N_G(y) - T| \leq 1$.

Case 1: $|N_G(y) \cap T| \geq 2$

By assumption $T \subseteq X \cup SMALL$ and so $y \in X$.

Case 2: $|N_G(y) \cap T| = 1$.

Then $d_G(y)=2$ implies $y \in X \cup W_1 \cup Y_2$.

■

Lemma 4.2

If c is large enough and G satisfies the conditions in Lemmas 2.1, 2.2 then H_2 is connected.

Proof

If $H=H_2$ is not connnected then there exists a *nonempty* $S \subseteq V(H)$ such that $N_H(S) = \emptyset$. We show that this is not possible for c large enough. (2.23) implies that $|S| \geq n/14$. (4.1) implies that, for c large, fewer than $2ce^{-c}n$ vertices are deleted from G in producing H. Then (2.2) implies that at most $8c^2e^{-c}n$ edges are lost in the construction. But then (2.6) implies that not all edges with one vertex in S have been deleted.

■

The analogue of Lemma 3.2 is

Lemma 4.3

Let H be a connected graph which is non-hamiltonian. Then

(a) (4.2) no edge of H joins the endpoints of any longest path of H.

(b) Let $U=\{u_1, u_2,\ldots,u_t\}$ be the set of vertices which are endpoints of longest paths of H. For i=1,2,...t there exists $U_i \subseteq U$ satisfying

(4.3a) $|N_H(U_i)| < 2\,|U_i|$;

(4.3b) $w \in U_i$ implies $\{u_i,w\} \notin E(H)$ and there is some longest path of H

that joins u_i to w.

Proof

(4.2) is straightforward and (4.3) is from Posá [11] ∎

We can now give an outline of the

Proof of Theorem 1.2

We define E^b, E^g and G^b as in the proof of Theorem 1.1 and let $H_2^b = H_2(G^b)$. Let now

$\mathcal{G} \equiv G = G_{n,p}$ satisfies the conditions of Lemma's 2.1, 2.2 and H_2 is not hamiltonian, which implies that (4.2) holds with $H=H_2$.

We have only to show that (3.5) holds with this definition of $\mathcal{G}$. Let now

$\mathcal{E} \equiv$ (a) $\emptyset \neq S \subseteq A_2(G^b)$, $|S| \le n/14$ implies $|N_{H_2^b}(S)| \ge 2\,|S|$;

(b) there does not exist $e=\{v,w\} \in E^b \cup E^g$ such that v, w are the endpoints of some longest path of H_2^b.

We replace (3.6) by

(4.3a) $\Pr(\mathcal{E} | \mathcal{G}) \ge (1-o(1))(1-p')^{3n/2}$;

(4.3b) $\Pr(\mathcal{E}) \le (1-p_1)^{n^2/392}$.

This will prove the theorem.

To prove (4.3a) let $G_0 \in \mathcal{G}$ be fixed and let P_0 be some longest path of H_2. We define $\mathcal{E}_1$, $\mathcal{E}_2$ as before and define $\mathcal{E}_3 \equiv P_0 \cap E^g = \emptyset$.

Now $\mathcal{E}_1 \cap \mathcal{E}_2$ implies that $A_2(G_0^b) = A_2(G_0)$ and then (3.8) and (4.3a) will

follow in the same way as (3.8) and (3.6a) previously.

To prove (4.3b) we use (3.11) and concentrate on the case where $H_2(\Gamma)$ satisfies $\mathcal{E}$(a). We note that for $\mathcal{E}$(b) to hold there is no $\{v,w\} \in E^g$, $v_i \in U$, $w \in U_i$ where U, U_1, $U_2,\ldots U_t$ are defined by (4.3) w.r.t. $H=H_2(\Gamma)$. (4.3b) follows from Remark 3.1 and $\mathcal{E}$(a) as before.

We note that if we put $c=\log n+\log\log n+\omega$ where $\omega\to\infty$ then we obtain the result of Komlós and Szemerédi [8]and Korsunov [9].

Finally note that our Corollary follows from the Percolation Theorem of McDiarmid [10].

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